Topological Loss and Gradient

Given two diagrams, Dgm(g) and Dgm(f), where g is the ground-truth function (not a variable), and f is the likelihood function. Dgm(g) has m dots at coordinates (0,1), corresponding to m 1D homology features created at time zero and killed at one (recall g is a binary function). Dgm(f) has n dots. One of them is at (0,1), corresponding to a homology we created at zero (by padding zeros around the image in the beginning). Among the other n-1 with persistence less than one, we sort them by persistence, and pick the top m-1, match each of them to one dot in Dgm(g), with coordinate (0,1). For the rest n-m dots, we match them to their nearest neighbor at the diagonal. The total topology loss is , in which is the dot in Dgm(g) matched to p. So the loss is measuring the distance from Dgm(f) to Dgm(g).

But it needs to be implemented on the likelihood function f. For dot p, we call bcp(p) and dcp(p) the birth critical point and death critical point of p. Same for dot . Then we have the point p’s coordinates being (f(bcp(p)), f(dcp(p))), and ’s coordinates being (g(bcp), g(dcp())). The loss is . Taking partial derivative with regard to f(bcp(p)), we get . And taking partial derivative with regard to f(dcp(p)), we get .

The force (-1 times the gradient) is the force on different pixels (see legends in the critical point figure, I forgot to multiply by 2 though. The force is to push each dot p towards within the diagram (shown in the diagram figure). Intuitively, for a dot we trying to push towards (0,1) (dot-to-fix). The force will decrease the value of its creating pixel with twice of its value, . For for the death critical point, the force increase its value with . Similarly, we have the force for the dots we are pushing towards the diagonal (I might miscalculate by a square root of 2, not sure).